arXiv:1006.5267v1 [math.DG] 28 Jun 2010

ON ALMOST ISOMETRY THEOREM IN ALEXANDROV SPACES WITH CURVATURE BOUNDED BELOW [1, 2]

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Abstract.

In this paper we give a new proof for an almost isometry theorem in Alexandrov spaces with curvature bounded below.

Key words. Alexandrov spaces, GH-approximation, almost isometry.

Due to the great work by Perel'man on Poincaré conjecture, Alexandrov geometry (especially with curvature bounded below) together with Gromov-Hausdorff convergence theory attracts more and more attentions.

A fundamental and significant work on Alexandrov spaces with curvature bounded below is of Burago-Gromov-Perel'man ([1]). One important result in [1] is an almost isometry theorem (see Theorem 0.1 below). We find a key lemma of its proof is incorrect (see "Lemma" 1.2 and Example 1.3 below). We suppose that the authors of [1] missed some condition. In the present paper we adjust the conditions of the lemma so that the conclusion of it still holds (see Lemma 2.1 below). Unfortunately, from the modified lemma the original proof of the theorem cannot go through. For this reason, we supply a new proof for the theorem in this paper.

0 Notations and main theorem

We first give some notations, which are almost copied from [1].

• $|xy|$ always denotes the distance between two points x and y in a metric space.

• For any three points p, q, r in a length space, we associate a triangle $\triangle\tilde{p}\tilde{q}\tilde{r}$ on the k-plane (2-dimensional complete and simply-connected Riemannian manifold of constant curvature k) with $|\tilde{p}\tilde{q}| = |pq|, |\tilde{p}\tilde{r}| = |pr|$ and $|\tilde{r}\tilde{q}| = |rq|$. For $k \leqslant 0$ and for $k > 0$ with $|pq| + |pr| + |qr| \leqslant 2\pi/\sqrt{k}$, such a triangle always exists. We denote by $\tilde{\angle}pqr$ the angle of the triangle $\triangle\tilde{p}\tilde{q}\tilde{r}$ at vertex $\tilde{q}$.

• M always denotes an Alexandrov space with curvature bounded below by k, which is a length space and in which there exists a neighborhood U_x around any $x \in M$ such that for any four (distinct) points $(a; b, c, d)$ in U_x

$$\tilde{\angle}bac + \tilde{\angle}bad + \tilde{\angle}cad \leqslant 2\pi.$$

[1] Supported by NSFC 10801011 and 10826052.

[2] Mathematics Subject Classification (2000): 53-C20.

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• A point $p \in M$ is called an (n,δ)-strained point if there are n pairs of points (a_i,b_i) distinct from p such that for $i \neq j$

$$\tilde{\angle} a_i p b_i > \pi - \delta,\ \tilde{\angle} a_i p a_j > \pi/2 - \delta,$$
$$\tilde{\angle} a_i p b_j > \pi/2 - \delta,\ \tilde{\angle} b_i p b_j > \pi/2 - \delta.$$

$\{(a_i,b_i)\}_{i=1}^n$ is called an (n,δ)-strainer at p (which is obviously a generalization of a coordinate frame). We say that the (n,δ)-strainer $\{(a_i,b_i)\}_{i=1}^n$ at p is R-long if $|a_i p| > \dfrac{R}{\delta}$ and $|b_i p| > \dfrac{R}{\delta}$ for all i. And we denote by $M(n,\delta,R)$ the set of points with R-long (n,δ)-strainer in M.

• An important fact is that if any neighborhood of a point $p \in M$ contains an (n,δ)-strained point (δ is sufficient small) but no $(n+1,\delta)$-strained point, then any neighborhood of any other point in M has the same property (see §6 in [1]). And it follows that the dimension of such M is defined to be n.

• We always denote by $\varkappa(\cdot)$ or $\varkappa(\cdot,\cdot)$ (resp. C) a positive function which is infinitesimal at zero (e.g. $\varkappa(\delta,\delta_1) \longrightarrow 0$ as $\delta,\delta_1 \longrightarrow 0$) (resp. a constant depending only on n); however we do not distinguish any two distinct $\varkappa$-functions with the same parameters (resp. any two such constants) when we use them.

• A map f between metric spaces (X,d_1) and (Y,d_2) is called a GH$_\epsilon$-approximation if $B_\epsilon(f(X)) = Y$ and $|d_2(f(x_1),f(x_2)) - d_1(x_1,x_2)| < \epsilon$ for any $x_1,x_2 \in X$.

• $f : (X,d_1) \longrightarrow (Y,d_2)$ is called a $\varkappa(\delta)$-*almost distance-preserving map* if

$$\left|1 - \frac{|f(x)f(y)|}{|xy|}\right| < \varkappa(\delta) \text{ for any } x,y \in X;$$

and if in addition f is a bijection, f is called a $\varkappa(\delta)$-*almost isometry*.

• We say that $\bar{f} : (X,d_1) \longrightarrow (Y,d_2)$ is ν-close to f if $|f(x)\bar{f}(x)| < \nu$ for any $x \in X$.

Now we formulate the almost isometry theorem in [1] mentioned at the beginning.

Theorem 0.1 (Theorem 9.8 in [1]) *Let M_1 and M_2 be two compact n-dimensional Alexandrov spaces with the same low curvature bound, and let $h : M_1 \to M_2$ be a GH$_\nu$-approximation. Then for sufficiently small δ and $\dfrac{\nu}{R\delta^3}$, there exists a $\varkappa(\delta, \frac{\nu}{R\delta^3})$-almost distance preserving map $\overline{h} : M_1(n,\delta,R) \to M_2$ which is $C\nu$-close to h.*

It is not difficult to conclude from Theorem 0.1 the following important corollary.

Corollary 0.2 ([1]) *In Theorem 0.1, if in addition each point of M_2 is (n,δ)-strained, then there exists a $\varkappa(\delta,\nu)$-almost isometry $\overline{h} : M_1 \longrightarrow M_2$ which is $C\nu$-close to h.*

Theorem 0.1 (or Corollary 0.2) plays an important role when one studies a converging sequence (with respect to the Gromov-Hausdorff distance defined by the GH-approximation) of n-dimensional Alexandrov spaces with the same low curvature bound.

In this paper we give the proof of the following sharper version of Theorem 0.1.

Theorem A *Let M_1 and M_2 be two compact n-dimensional Alexandrov spaces with the same low curvature bound, and let $h : M_1 \longrightarrow M_2$ be a GH$_\nu$-approximation. Then for sufficiently small δ and $\nu < \delta^2 R$, there exists a $\varkappa(\delta)$-almost distance preserving map $\overline{h} : M_1(n,\delta,R) \longrightarrow M_2$ which is $C\nu$-close to h.*

The construction of $\bar{h}$ is almost copied from [1] (see Section 3). The main difference between our proof and the proof of Theorem 0.1 in [1] is how to verify that $\bar{h}$ almost preserves distance (see Section 4). Of course, we use some ideas and results in [1].

Remark 0.3 In [2] Yamaguchi proved that, without the assumption of the dimension of M_1, there is an almost Lipschitz submersion from M_1 to M_2 if each point of M_2 is (n, δ)-strained in Theorem 0.1. This result (which appears as a conjecture in [1]) coincides with Corollary 0.2 if the dimension of M_1 is n. The key approach to construct the almost Lipschitz submersion in [2] is to embed an Alexandrov space with curvature bound below M into $L^2(M)$. Compared with it, the base of the construction of $\bar{h}$ of Theorem 0.1 (or A) is that $M_1(n, \delta, R)$ is locally almost isometric to the n-dimensional Euclidean space (see Theorem 1.1 below).

1 Center of mass and a key lemma in [1]

The main tool in the construction of $\bar{h}$ ([1]) in Theorem 0.1 (or A) is "center of mass". Recall that the center of mass of a set of points $Q = \{q_1, q_2, \cdots, q_l\} \subset \mathbb{R}^n$ with weights $W = (w_1, w_2, \cdots, w_l)$ (where $\sum_{j=1}^{l} w_j = 1$ and $w_j \geqslant 0$) is defined to be

$$Q_W = \sum_{j=1}^{l} w_j q_j.$$

The construction of the center of mass for a set of points in M is based on the following important result.

Theorem 1.1 (Theorem 9.4 in [1]) *Let M be an n-dimensional Alexandrov space with curvature bounded below, and let $\{(a_i, b_i)\}_{i=1}^{n}$ be an (n, δ)-strainer at $p \in M$. Then there exist neighborhoods U and V around p and $(|a_1p|, |a_2p|, \cdots, |a_np|) \in \mathbb{R}^n$ respectively such that*

$$f : U \longrightarrow V \subset \mathbb{R}^n \text{ given by } f(q) = (|a_1q|, |a_2q|, \cdots, |a_nq|)$$

is a $\varkappa(\delta, \delta_1)$-almost isometry, where $\delta_1 = \max\limits_{1 \leqslant i \leqslant n} \{|pa_i|^{-1}, |pb_i|^{-1}\} \cdot \operatorname{diam} U.$

If $Q = \{q_1, q_2, \cdots, q_l\}$ belongs to U in Theorem 1.1, and if in addition $f(U)$ is convex in $\mathbb{R}^n$, then the center of mass of Q with weights W is defined to be ([1])

$$Q_W = f^{-1}\left(\sum_{j=1}^{l} w_j f(q_j)\right).$$

Obviously, Q_W depends on the choice of the (n, δ)-strainer at p.

Now we give the key lemma in [1] mentioned at the beginning of Section 0, which plays a crucial role in verifying that $\bar{h}$ in Theorem 0.1 almost preserves distance.

"Lemma" 1.2 *Let $p, U, \{(a_i, b_i)\}_{i=1}^n$ and f be the same as in Theorem 1.1, and let $\{(s_i, t_i)\}_{i=1}^n$ be another (n, δ)-strainer at p with*

$$\delta_1 = \max\left\{\frac{\operatorname{diam} U}{\min_i\{|pa_i|, |pb_i|\}}, \frac{\max_i\{|pa_i|, |pb_i|\}}{\min_i\{|ps_i|, |pt_i|\}}\right\}.$$

And let $Q = \{q_1, \cdots, q_l\}$ and $R = \{r_1, \cdots, r_l\}$ be two sets of points in U with

$$\max_j\{|q_j r_j|\} < (1+\delta)\min_j\{|q_j r_j|\}, \text{ and } |\max_j \tilde{\angle} s_i q_j r_j - \min_j \tilde{\angle} s_i q_j r_j| < \delta \text{ for } i = 1, \cdots n.$$

Assume that $f(U)$ is convex in $\mathbb{R}^n$. Then for any weights W^1 and W^2 such that $||W^1 - W^2|| < \delta_1$, the centers of mass Q_{W^1} and R_{W^2} (with respect to the strainer $\{(a_i, b_i)\}$) satisfy that

$$\left|1 - \frac{|q_j r_j|}{|Q_{W^1} R_{W^2}|}\right| < \varkappa(\delta, \delta_1)$$

and $|\tilde{\angle} s_i q_j r_j - \tilde{\angle} s_i Q_{W^1} R_{W^2}| < \varkappa(\delta, \delta_1)$ for $j = 1, \cdots, l$ and $i = 1, \cdots, n$.

Due to the following counterexample, we don't think that this lemma is correct.

Example 1.3 1. In fact, if $q_j = r_j$ for $j = 1, \cdots, l$ and $W^1 \neq W^2$, then $Q_{W^1} \neq R_{W^2}$ and thus

$$\left|1 - \frac{|q_j r_j|}{|Q_{W^1} R_{W^2}|}\right| = |1 - 0| > \varkappa(\delta, \delta_1).$$

2. If $|q_j r_j| \ll ||W^1 - W^2||$ for all j, then "$\left|1 - \frac{|q_j r_j|}{|Q_{W^1} R_{W^2}|}\right| < \varkappa(\delta, \delta_1)$" does not hold.

Inspired by the example, we add some stronger restriction on the weights W^1 and W^2 (see Lemma 2.1 below) so that the conclusion in "Lemma 1.2" still holds.

2 Modified key lemma

In this section we give a modified version of "Lemma" 1.2 which is formulated as follows (for convenience we divide it into two parts).

Lemma 2.1 *Let $p, U, \{(a_i, b_i)\}_{i=1}^n$ and f be the same as in Theorem 1.1, and let $\{(s_i, t_i)\}_{i=1}^n$ be another (n, δ)-strainer at p with*

$$\max\left\{\frac{\operatorname{diam} U}{\min_i\{|pa_i|, |pb_i|\}}, \frac{\max_i\{|pa_i|, |pb_i|\}}{\min_i\{|ps_i|, |pt_i|\}}\right\} < \delta.$$

And let $Q = \{q_1, \cdots, q_l\}$ and $R = \{r_1, \cdots, r_l\}$ be two sets of points in U. Then the following conclusions hold.

(2.1.1) The following statements are equivalent:

(1) $|\tilde{\angle} a_i q_j r_j - \tilde{\angle} a_i q_{j'} r_{j'}| < \varkappa(\delta)$ *for* $i = 1, \cdots, n$;

(2) $|\tilde{\angle} s_i q_j r_j - \tilde{\angle} s_i q_{j'} r_{j'}| < \varkappa(\delta)$ *for* $i = 1, \cdots, n$.

(2.1.2) Assume that $f(U)$ is convex in $\mathbb{R}^n$, and assume that

$$\max_j\{|q_j r_j|\} < (1 + \varkappa(\delta))\min_j\{|q_j r_j|\} \text{ and} \tag{2.1}$$

$$|\max_j \tilde{\angle} a_i q_j r_j - \min_j \tilde{\angle} a_i q_j r_j| < \varkappa(\delta) \text{ for } i = 1, \cdots, n. \tag{2.2}$$

Then for any weights W^1 and W^2 with $||W^1 - W^2|| \cdot \max_{j,j'} |r_j r_{j'}| < \varkappa(\delta) \min_j |q_j r_j|$, the centers of mass Q_{W^1} and R_{W^2} (with respect to the strainer $\{(a_i, b_i)\}$) satisfy that

$$\left|1 - \frac{|q_j r_j|}{|Q_{W^1} R_{W^2}|}\right| < \varkappa(\delta) \text{ and}$$

$$|\tilde{\angle} a_i q_j r_j - \tilde{\angle} a_i Q_{W^1} R_{W^2}| < \varkappa(\delta) \text{ for } j = 1, \cdots, l \text{ and } i = 1, \cdots, n.$$

(2.1.1) is proved in [1] (for convenience of readers we give its proof in Appendix). For the proof of (2.1.2) we need Lemmas 2.2 and 2.4.

Lemma 2.2 *Let $Q = \{q_1, q_2, \cdots, q_l\}$ and $R = \{r_1, r_2, \cdots, r_l\}$ be two sets of points in $\mathbb{R}^n$, and let $W^i = (w_1^i, w_2^i, \cdots, w_l^i)$ be two weights with $i = 1, 2$. Then*

$$\overrightarrow{Q_{W^1} R_{W^2}} = \sum_{j=1}^{l} w_j^1 \overrightarrow{q_j r_j} + \sum_{j=1}^{l} (w_j^2 - w_j^1) \overrightarrow{r_{j_0} r_j},$$

for any $j_0 \in \{1, 2, \cdots, l\}$.

Proof. Straightforward computation gives

$$\begin{aligned}
\overrightarrow{Q_{W^1} R_{W_2}} &= \sum_{j=1}^{l} w_j^2 r_j - \sum_{j=1}^{l} w_j^1 q_j \\
&= \sum_{j=1}^{l} w_j^1 (r_j - q_j) + \sum_{j=1}^{l} (w_j^2 - w_j^1) r_j \\
&= \sum_{j=1}^{l} w_j^1 \overrightarrow{q_j r_j} + \sum_{j=1}^{l} (w_j^2 - w_j^1) r_j - \sum_{j=1}^{l} (w_j^2 - w_j^1) r_{j_0} \\
&= \sum_{j=1}^{l} w_j^1 \overrightarrow{q_j r_j} + \sum_{j=1}^{l} (w_j^2 - w_j^1) \overrightarrow{r_{j_0} r_j}.
\end{aligned}$$

□

To simplify further considerations, we use the following definition.

Definition 2.3 For sets of points $Q = \{q_1, q_2\}$ and $R = \{r_1, r_2\}$ in $\mathbb{R}^n$, we say that $\overrightarrow{q_1 r_1}$ is $\varkappa(\delta)$-*almost parallel* to $\overrightarrow{q_2 r_2}$ if

$$\angle(\overrightarrow{q_1 r_1}, \overrightarrow{q_2 r_2}) < \varkappa(\delta);$$

and if in addition

$$\left|1 - \frac{|q_1 r_1|}{|q_2 r_2|}\right| < \varkappa(\delta),$$

we say that $\overrightarrow{q_1 r_1}$ is $\varkappa(\delta)$-*almost equal* to $\overrightarrow{q_2 r_2}$.

Lemma 2.4 *Let $p, U, \{(a_i, b_i)\}_{i=1}^n$ and f be the same as in Lemma 2.1. Then for any points $x_1, x_2, y_1, y_2 \in U$, the following statements are equivalent:*
(1) $|\tilde{\angle} a_i x_1 y_1 - \tilde{\angle} a_i x_2 y_2| < \varkappa(\delta)$ *for* $i = 1, 2, \cdots, n$;
(2) $\overrightarrow{f(x_1)f(y_1)}$ *is* $\varkappa(\delta)$*-almost parallel to* $\overrightarrow{f(x_2)f(y_2)}$.

Lemma 2.4 is implied in [1] (we will give its proof in Appendix).

Proof of (2.1.2). According to Theorem 1.1 and Lemma 2.4, inequalities (2.1) and (2.2) imply that $\overrightarrow{f(q_j)f(r_j)}$ are $\varkappa(\delta)$-almost equal each other for $j = 1, 2, \cdots, l$. Therefore it follows from Lemma 2.2 that $\overrightarrow{f(Q_{W^1})f(R_{W^2})}$ is $\varkappa(\delta)$-almost equal to $\overrightarrow{f(q_j)f(r_j)}$ for every j (note that $f(Q_{W^1}) = \sum_{j=1}^{l} w_j^1 f(q_j)$ and $f(R_{W^2}) = \sum_{j=1}^{l} w_j^2 f(r_j)$, and $||W^1 - W^2|| \cdot \max_{j,j'} |r_j r_{j'}| < \varkappa(\delta) \min_j |q_j r_j|$). And thus the conclusion in (2.1.2) follows from Lemma 2.4 and the fact that f is a $\varkappa(\delta)$-almost isometry. $\square$

At the end of this section we give a corollary of (2.1.1), which will be used in gluing local almost isometries to a global one (see next section).

Corollary 2.5 *Let $p, U, \{(a_i, b_i)\}_{i=1}^n$ and $\{(s_i, t_i)\}_{i=1}^n$ be the same as in Lemma 2.1. Let $\{(a_i', b_i')\}_{i=1}^n$ be an (n, δ)-strainer at another point p', and let U' be a neighborhood around p determined by Theorem 1.1 (with respect to $\{(a_i', b_i')\}$). Moreover we assume that $\{(s_i, t_i)\}_{i=1}^n$ is also an (n, δ)-strainer at p', and*

$$\left\{ \frac{\operatorname{diam} U'}{\min_i \{|p'a_i'|, |p'b_i'|\}}, \frac{\max_i \{|p'a_i'|, |p'b_i'|\}}{\min_i \{|p's_i|, |p't_i|\}} \right\} < \delta.$$

Then for any points $x_1, x_2, y_1, y_2 \in U_1 \cap U_2$, the following statements are equivalent:
(1) $|\tilde{\angle} a_i x_1 y_1 - \tilde{\angle} a_i x_2 y_2| < \varkappa(\delta)$ *for* $i = 1, \cdots, n$;
(2) $|\tilde{\angle} a_i' x_1 y_1 - \tilde{\angle} a_i' x_2 y_2| < \varkappa(\delta)$ *for* $i = 1, \cdots, n$.

3 The construction of $\bar{h}$ in Theorem A

In this section, we give the construction of the map $\bar{h}$ in Theorem A, which is almost copied from [1].

Since the closure of $M_1(n, \delta, R)$ is compact, we can select $x_j \in M_1(n, \delta, R)$ with $j = 1, \cdots, N_1$ such that

$$\bigcup_{j=1}^{N_1} B_{x_j}(\delta R) \supset \bigcup_{j=1}^{N_1} B_{x_j}(\frac{1}{3}\delta R) \supset M_1(n, \delta, R). \tag{3.1}$$

Without loss of generality, we can assume that the multiplicity of the cover $\{B_{x_j}(\delta R)\}$ is bounded by a number N depending only on the dimension n (see Theorem 1.1 for the dimension).

Since $x_j \in M_1(n, \delta, R)$, there exists an R-long (n, δ)-strainer $\{(s_i^j, t_i^j)\}_{i=1}^n$ at x_j (with $\min_i \{|x_j s_i^j|, |x_j t_i^j|\} > \frac{R}{\delta}$), and thus there exists a δR-long (n, δ)-strainer $\{(a_i^j, b_i^j)\}_{i=1}^n$

at x_j (with $\min_i\{|x_j a_i^j|, |x_j b_i^j|\} > R$) such that

$$\frac{\max_i\{|x_j a_i^j|, |x_j b_i^j|\}}{\min_i\{|x_j s_i^j|, |x_j t_i^j|\}} < \delta.$$

Denote by f_j and U_j the associated map and the neighborhood around x_j in Theorem 1.1 with respect to the strainer $\{(a_i^j, b_i^j)\}$. Moreover we select U_j such that $f_j(U_j)$ is convex in $\mathbb{R}^n$; and such that

$$B_{x_j}(2\delta R/3) \subset U_j \subset B_{x_j}(\delta R) \tag{3.2}$$

which implies that

$$f_j\big|_{U_j} \text{ is a } \varkappa(\delta)\text{-almost isometry (see Theorem 1.1).}$$

Since h is a GH_ν-approximation with $\nu < R\delta^2$, $\{(h(a_i^j), h(b_i^j))\}_{i=1}^n$ and $\{(h(s_i^j), h(t_i^j))\}_{i=1}^n$ are $(n, 2\delta)$-strainers at $h(x_j)$. We consider the associated map g_j around $h(x_j)$ in Theorem 1.1 with respect to the strainer $\{(h(a_i^j), h(b_i^j))\}$, and we have that

$$g_j^{-1}\big|_{f_j(U_j)} \text{ is a } \varkappa(\delta)\text{-almost isometry.}$$

Obviously,

$$h_j = g_j^{-1} \circ f_j \text{ is a } \varkappa(\delta)\text{-almost isometry on each } U_j,$$

and for any $x \in U_j$

$$\begin{aligned}
&|h_j(x)h(x)| = (1 + \varkappa(\delta))|g_j(h_j(x))g_j(h(x))| \\
=&(1 + \varkappa(\delta))|f_j(x)g_j(h(x))| \\
=&(1 + \varkappa(\delta))\sqrt{(|a_1^j x| - |h(a_1^j)h(x)|)^2 + \cdots + (|a_n^j x| - |h(a_n^j)h(x)|)^2} \\
<&(1 + \varkappa(\delta))\sqrt{n}\nu \quad \text{(note that } h \text{ is a } \mathrm{GH}_\nu\text{-approximation)},
\end{aligned}$$

i.e. each h_j is $C\nu$-close to h on U_j.

We will use center of mass to glue all these local almost isometries h_j to a global one. We first define weight functions[4] $\phi_j : M_1 \longrightarrow \mathbb{R}$ by

$$\phi_j(x) = \begin{cases} 1 - \frac{2|xx_j|}{\delta R}, & x \in B_{x_j}(\delta R/2), \\ 0, & x \in M_1 \backslash B_{x_j}(\delta R/2). \end{cases}$$

Then for an arbitrary point $z \in M_1(n, \delta, R)$ we define a sequence $\{z_j\}_{j=1}^{N_1} \subset M_2$:

$$z_j = \begin{cases} g_j^{-1}\left(\dfrac{\Sigma_{j-1}(z)}{\Sigma_j(z)} g_j(z_{j-1}) + \dfrac{\phi_j(z)}{\Sigma_j(z)} g_j(h_j(z))\right) & z \in U_j \\ z_{j-1}, & z \notin U_j \end{cases},$$

[4]The original definition in [1] is $\phi_j(x) = (1 - 2|xx_j|/(\delta R))^N$ if $x \in B_{x_j}(\delta R/2)$, but we find that power 1 is sufficient. A basic reason for this is that we only need Lipschitz condition.

where $z_0=h(z)$, $\Sigma_0(z)=0$ and $\Sigma_j(z)=\sum\limits_{l=1}^{j}\phi_l(z)$ for $j\geqslant 1$. A basic fact is that

$$\Sigma_{N_1}(z)>\frac{1}{3}\ \text{ (see (3.1)).} \tag{3.3}$$

Now we define the desired map $\bar{h}: M_1(n,\delta,R)\longrightarrow M_2$ in Theorem A by

$$\bar{h}(z)=z_{N_1} \text{ for any } z\in M_1(n,\delta,R).$$

Since each h_j is $C\nu$-close to h, it is easy to see that

$$|h_j(z)h_{j'}(z)|<C\nu \text{ and } |z_jh_{j'}(z)|<C\nu, \tag{3.4}$$

and thus

$$\bar{h} \text{ is } C\nu\text{-close to } h.$$

In next section we will **verify that $\bar{h}$ $\varkappa(\delta)$-almost preserves distance**.

4 Verifying that $\bar{h}$ almost preserves distance

In this section, we verify that $\bar{h}$ constructed in Section 3 almost preserves distance, i.e. for any $y,z\in M_1(n,\delta,R)$,

$$\left|1-\frac{|\overline{h}(y)\overline{h}(z)|}{|yz|}\right|<\varkappa(\delta) \text{ or } \big||\overline{h}(y)\overline{h}(z)|-|yz|\big|<\varkappa(\delta)|yz|, \tag{4.1}$$

and thus **the proof of Theorem A is completed**.

We first observe that we only need to consider the case "$|yz|<R\delta^{3/2}$". In fact, if $|yz|\geqslant R\delta^{3/2}$, then $||\overline{h}(y)\overline{h}(z)|-|yz||<C\nu<CR\delta^2<|yz|\varkappa(\delta)$ (i.e. (4.1) holds) because $\overline{h}$ is $C\nu$-close to h which is a GH_ν-approximation.

Without loss of generality, we assume that $\phi_j(y)+\phi_j(z)\neq 0$ for $1\leqslant j\leqslant N_2$, but $\phi_j(y)+\phi_j(z)=0$ for $N_2<j\leqslant N_1$. Note that if $\phi_j(y)\neq 0$ (i.e., $y\in B_{x_j}(\delta R/2)$), then $z\in B_{x_j}(\delta 2R/3)\subset U_j$ (see (3.2)) because $|yz|<R\delta^{3/2}$ (δ is sufficient small). Then

$$y,z\in U_j \text{ for } j=1,\cdots,N_2 \text{ and } y,z\notin U_j \text{ for } j>N_2,$$

which implies that $N_2\leqslant N$ (a number depending on n) and that

$$\bar{h}(y)=y_{N_2} \text{ and } \bar{h}(z)=z_{N_2}. \tag{4.2}$$

And we can define two new sequences $\{\overline{y}_j\}_{j=1}^{N_2}$ and $\{\overline{z}_j\}_{j=1}^{N_2}$ in M_2 (which are not introduced in [1]):

$$\overline{y}_j=g_j^{-1}\left(\sum_{l=1}^{j}\frac{\phi_l(y)}{\Sigma_j(y)}g_j(h_l(y))\right) \text{ and } \overline{z}_j=g_j^{-1}\left(\sum_{l=1}^{j}\frac{\phi_l(z)}{\Sigma_j(z)}g_j(h_l(z))\right).$$

Note that

$$\overline{y}_j=y_j \text{ and } \overline{z}_j=z_j \text{ for } j=1,2. \tag{4.3}$$

Now we give two claims.

Claim 1[5]:

$$\big||\overline{y}_{N_2}\overline{z}_{N_2}| - |yz|\big| < \varkappa(\delta)|yz|.$$

Claim 2:

$$||y_{N_2}z_{N_2}| - |\overline{y}_{N_2}\overline{z}_{N_2}|| < \varkappa(\delta)|yz|.$$

Obviously, **Claims 1** and **2** (together with (4.2)) imply (4.1). Hence **we only need to verify Claims 1 and 2**.

• **The proof of Claim 1**:

Note that $\overline{y}_{N_2}$ (resp. $\overline{z}_{N_2}$) is the center of mass of $\{h_j(y)\}_{j=1}^{N_2}$ (resp. $\{h_j(z)\}_{j=1}^{N_2}$) with weights $W_y = (\frac{\phi_1(y)}{\Sigma_{N_2}(y)}, \cdots, \frac{\phi_{N_2}(y)}{\Sigma_{N_2}(y)})$ (resp. $W_z = (\frac{\phi_1(z)}{\Sigma_{N_2}(z)}, \cdots, \frac{\phi_{N_2}(z)}{\Sigma_{N_2}(z)})$) with respect to the (n,δ)-strainer $\{(h(a_i^{N_2}), h(b_i^{N_2}))\}_{i=1}^n$ at $h(x_{N_2})$. Then according to (2.1.2), **Claim 1** follows from the following three properties.

(i) Since each h_j is a $\varkappa(\delta)$-almost isometry, we have

$$\max_j\{|h_j(y)h_j(z)|\} < (1+\varkappa(\delta))\min_j\{|h_j(y)h_j(z)|\}. \tag{4.4}$$

(ii) For any fixed j,

$$|\max_l \tilde\angle h(a_i^j)h_l(y)h_l(z) - \min_l \tilde\angle h(a_i^j)h_l(y)h_l(z)| < \varkappa(\delta) \text{ for } i = 1, \cdots, n. \tag{4.5}$$

This is proved in [1] (we give its proof in Appendix in which the strainers $\{(s_i^j, t_i^j)\}$ will be used).

(iii)

$$||W_y - W_z|| \cdot \max_{j,j'} |h_j(z)h_{j'}(z)| < \varkappa(\delta)\min_j |h_j(y)h_j(z)|. \tag{4.6}$$

In order to prove inequality (4.6), we first give an estimate

$$\left|\frac{\phi_l(y)}{\Sigma_j(y)} - \frac{\phi_l(z)}{\Sigma_j(z)}\right| \leqslant \frac{C|yz|}{\delta R \Sigma_j(y)} \text{ for } 1 \leqslant l \leqslant j \leqslant N_2. \tag{4.7}$$

In fact, for any $1 \leqslant l \leqslant N_2$ we have $|\phi_l(y) - \phi_l(z)| = 2\frac{||zx_l| - |yx_l||}{\delta R} \leqslant \frac{2|yz|}{\delta R}$, and thus

$$\begin{aligned}\left|\frac{\phi_l(y)}{\Sigma_j(y)} - \frac{\phi_l(z)}{\Sigma_j(z)}\right| =& \frac{1}{\Sigma_j(y)}\left|\phi_l(y) - \frac{\phi_l(z)\Sigma_j(y)}{\Sigma_j(z)}\right| \\ =& \frac{1}{\Sigma_j(y)}\left|\phi_l(y) - \phi_l(z) - \phi_l(z)\frac{\Sigma_j(y)-\Sigma_j(z)}{\Sigma_j(z)}\right| \\ \leqslant& \frac{1}{\Sigma_j(y)}\max_l\{|\phi_l(y)-\phi_l(z)|\}\cdot N_2 \\ \leqslant& \frac{C|yz|}{\delta R\Sigma_j(y)}.\end{aligned}$$

Note that inequality (4.6) follows from (4.7), $\Sigma_{N_2}(y) > \frac{1}{3}$ (see (3.3)) and $|h_j(z)h_{j'}(z)| < C\nu < CR\delta^2$ (see (3.4)).

[5]The present proof is mainly inspired by this observation.

• **The proof of Claim 2**:

Put $\overrightarrow{\alpha}_j = \overrightarrow{g_j(y_j)g_j(z_j)} - \overrightarrow{g_j(\overline{y}_j)g_j(\overline{z}_j)}$, $j = 1, \cdots, N_2$. Since each g_j is a $\varkappa(\delta)$-almost isometry, **Claim 2** is equivalent to

$$|\overrightarrow{\alpha}_{N_2}| < \varkappa(\delta)|yz|. \tag{4.8}$$

Subclaim:

$$|\overrightarrow{\alpha}_j| \leqslant \frac{C|yz|\nu}{\delta R\Sigma_j(y)} + \frac{\Sigma_{j-1}(y)}{\Sigma_j(y)}(1+\varkappa(\delta))|\overrightarrow{\alpha}_{j-1}| + \varkappa(\delta)|yz| \text{ for } j = 2, \cdots, N_2. \tag{4.9}$$

It follows from the subclaim that

$$\begin{aligned}
|\overrightarrow{\alpha}_{N_2}| &\leqslant \frac{C|yz|\nu}{\delta R\Sigma_{N_2}(y)} + \varkappa(\delta)|yz| + \frac{\Sigma_{N_2-1}(y)}{\Sigma_{N_2}(y)}(1+\varkappa(\delta))|\overrightarrow{\alpha}_{N_2-1}| \\
&\leqslant \frac{C|yz|\nu}{\delta R\Sigma_{N_2}(y)} + \varkappa(\delta)|yz| + \frac{\Sigma_{N_2-2}(y)}{\Sigma_{N_2}(y)}(1+\varkappa(\delta))|\overrightarrow{\alpha}_{N_2-2}| \\
&\leqslant \cdots \\
&\leqslant \frac{C|yz|\nu}{\delta R\Sigma_{N_2}(y)} + \varkappa(\delta)|yz| + \frac{\Sigma_2(y)}{\Sigma_{N_2}(y)}(1+\varkappa(\delta))|\overrightarrow{\alpha}_2| \\
&< \varkappa(\delta)|yz| \text{ (note that } \Sigma_{N_2}(y) > \frac{1}{3},\ \nu < R\delta^2 \text{ and } |\overrightarrow{\alpha}_2| = 0 \text{ (see (4.3)))}.
\end{aligned}$$

Now we only need to verify the subclaim.

To simplify notations in the following computations, we let $\tilde{x}$ denote $g_j(x)$ for any $x \in U_j$.

Recall that

$$\tilde{y}_j = \frac{\Sigma_{j-1}(y)}{\Sigma_j(y)}\tilde{y}_{j-1} + \frac{\phi_j(y)}{\Sigma_j(y)}\widetilde{h_j(y)} \text{ and } \tilde{\overline{y}}_j = \sum_{l=1}^{j}\frac{\phi_l(y)}{\Sigma_j(y)}\widetilde{h_l(y)}$$

($\tilde{z}_j$ and $\tilde{\overline{z}}_j$ have the same form respectively). Through straightforward computation, one can get

$$\overrightarrow{\alpha}_j = \frac{\Sigma_{j-1}(y)}{\Sigma_j(y)}\left(\overrightarrow{\tilde{\overline{y}}_{j-1}\tilde{\overline{z}}_{j-1}} - \sum_{l=1}^{j-1}\frac{\phi_l(y)}{\Sigma_{j-1}(y)}\overrightarrow{\widetilde{h_l(y)}\widetilde{h_l(z)}}\right) + \sum_{l=1}^{j-1}\left(\frac{\phi_l(z)}{\Sigma_j(z)} - \frac{\phi_l(y)}{\Sigma_j(y)}\right)\overrightarrow{\widetilde{h_l(z)}\tilde{\overline{z}}_{j-1}}$$

Put $\overrightarrow{\beta} = \overrightarrow{\tilde{\overline{y}}_{j-1}\tilde{\overline{z}}_{j-1}} - \sum\limits_{l=1}^{j-1}\frac{\phi_l(y)}{\Sigma_{j-1}(y)}\overrightarrow{\widetilde{h_l(y)}\widetilde{h_l(z)}}$ and $\overrightarrow{\gamma} = \sum\limits_{l=1}^{j-1}\left(\frac{\phi_l(z)}{\Sigma_j(z)} - \frac{\phi_l(y)}{\Sigma_j(y)}\right)\overrightarrow{\widetilde{h_l(z)}\tilde{\overline{z}}_{j-1}}$, and thus

$$\overrightarrow{\alpha}_j = \frac{\Sigma_{j-1}(y)}{\Sigma_j(y)}\overrightarrow{\beta} + \overrightarrow{\gamma}. \tag{4.10}$$

It follows from inequalities (4.7) and (3.4) that

$$|\overrightarrow{\gamma}| \leqslant \sum_{l=1}^{j-1}\frac{C|yz|}{R\delta\Sigma_j(y)} \cdot C\nu \leqslant \frac{C|yz|\nu}{R\delta\Sigma_j(y)}. \tag{4.11}$$

In order to estimate $|\overrightarrow{\beta}|$, we introduce two points $\overline{z}'_{j-1}$ and z'_{j-1} such that

$$\overline{z}'_{j-1} = g_{j-1}^{-1}\left(\sum_{l=1}^{j-1}\frac{\phi_l(y)}{\Sigma_{j-1}(y)}g_{j-1}(h_l(z))\right)$$

and

$$\overrightarrow{g_{j-1}(y_{j-1})g_{j-1}(z'_{j-1})} = \overrightarrow{g_{j-1}(\overline{y}_{j-1})g_{j-1}(\overline{z}_{j-1})}. \tag{4.12}$$

Now we put

$$\begin{aligned}
\overrightarrow{\beta}^1 &= \overrightarrow{\widetilde{y}_{j-1}\widetilde{z}_{j-1}} - \overrightarrow{\widetilde{y}_{j-1}\widetilde{z'}_{j-1}},\\
\overrightarrow{\beta}^2 &= \overrightarrow{\widetilde{y}_{j-1}\widetilde{z'}_{j-1}} - \overrightarrow{\widetilde{\overline{y}}_{j-1}\widetilde{\overline{z}}_{j-1}},\\
\overrightarrow{\beta}^3 &= \overrightarrow{\widetilde{\overline{y}}_{j-1}\widetilde{\overline{z}}_{j-1}} - \overrightarrow{\widetilde{\overline{y}}_{j-1}\widetilde{\overline{z}'}_{j-1}},\\
\overrightarrow{\beta}^4 &= \overrightarrow{\widetilde{\overline{y}}_{j-1}\widetilde{\overline{z}'}_{j-1}} - \sum_{l=1}^{j-1}\frac{\phi_l(y)}{\Sigma_{j-1}(y)}\overrightarrow{\widetilde{h_l(y)}\widetilde{h_l(z)}}.
\end{aligned}$$

Obviously $\overrightarrow{\beta} = \overrightarrow{\beta}^1 + \overrightarrow{\beta}^2 + \overrightarrow{\beta}^3 + \overrightarrow{\beta}^4$.

Firstly,

$$\begin{aligned}
|\overrightarrow{\beta}^1| = \left|\overrightarrow{\widetilde{z'}_{j-1}\widetilde{z}_{j-1}}\right| &= (1+\varkappa(\delta))|z'_{j-1}z_{j-1}|\\
&=(1+\varkappa(\delta))\left|\overrightarrow{g_{j-1}(z'_{j-1})g_{j-1}(z_{j-1})}\right|\\
&=(1+\varkappa(\delta))\left|\overrightarrow{g_{j-1}(y_{j-1})g_{j-1}(z_{j-1})} - \overrightarrow{g_{j-1}(y_{j-1})g_{j-1}(z'_{j-1})}\right|\\
\text{(by (4.12))} &=(1+\varkappa(\delta))\left|\overrightarrow{g_{j-1}(y_{j-1})g_{j-1}(z_{j-1})} - \overrightarrow{g_{j-1}(\overline{y}_{j-1})g_{j-1}(\overline{z}_{j-1})}\right|\\
&=(1+\varkappa(\delta))|\overrightarrow{\alpha}_{j-1}|.
\end{aligned}$$

Secondly,

$$\begin{aligned}
|\overrightarrow{\beta}^3| = \left|\overrightarrow{\widetilde{\overline{z}'}_{j-1}\widetilde{\overline{z}}_{j-1}}\right| &= (1+\varkappa(\delta))|\overline{z}'_{j-1}\overline{z}_{j-1}|\\
&=(1+\varkappa(\delta))|\overrightarrow{g_{j-1}(\overline{z}'_{j-1})g_{j-1}(\overline{z}_{j-1})}|\\
&=(1+\varkappa(\delta))\left|\sum_{l=1}^{j-1}\left(\frac{\phi_l(y)}{\Sigma_{j-1}(y)} - \frac{\phi_l(z)}{\Sigma_{j-1}(z)}\right)g_{j-1}(h_l(z))\right|\\
&=(1+\varkappa(\delta))\left|\sum_{l=1}^{j-1}\left(\frac{\phi_l(y)}{\Sigma_{j-1}(y)} - \frac{\phi_l(z)}{\Sigma_{j-1}(z)}\right)\overrightarrow{g_{j-1}(h_1(z))g_{j-1}(h_l(z))}\right|\\
&\leqslant\frac{C|yz|\nu}{\delta R\Sigma_{j-1}(y)} \quad \text{(similar to getting (4.11))}.
\end{aligned}$$

Thirdly, we estimate $|\overrightarrow{\beta}^4|$. According to Lemma 2.4, it follows from (4.4) and (4.5) that for any $1 \leqslant l, l_1, l_2 \leqslant N_2$

$$\overrightarrow{g_l(h_{l_1}(y))g_l(h_{l_1}(z))} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{g_l(h_{l_2}(y))g_l(h_{l_2}(z))}, \tag{4.13}$$

and thus

$$\overrightarrow{g_{j-1}(\overline{y}_{j-1})g_{j-1}(\overline{z}'_{j-1})} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{g_{j-1}(h_l(y))g_{j-1}(h_l(z))}.$$

Then according to Corollary 2.5[6] and Lemma 2.4,

$$\overrightarrow{\tilde{\overline{y}}_{j-1}\tilde{\overline{z}}'_{j-1}} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{\widetilde{h_l(y)}\widetilde{h_l(z)}}. \tag{4.14}$$

On the other hand, by (4.13)

$$\sum_{l=1}^{j-1}\frac{\phi_l(y)}{\Sigma_{j-1}(y)}\overrightarrow{\widetilde{h_l(y)}\widetilde{h_l(z)}} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{\widetilde{h_l(y)}\widetilde{h_l(z)}}.$$

Therefore it follows that

$$|\overrightarrow{\beta}^4| < \varkappa(\delta)|\widetilde{h_l(y)}\widetilde{h_l(z)}| = \varkappa(\delta)|yz|.$$

Finally, we estimate $|\overrightarrow{\beta}^2|$. Note that it follows from (4.14) that $|\overrightarrow{\tilde{\overline{y}}_{j-1}\tilde{\overline{z}}'_{j-1}}| < \varkappa(\delta)|yz|$, and thus

$$|\overrightarrow{\tilde{\overline{y}}_{j-1}\tilde{\overline{z}}_{j-1}}| \leqslant |\overrightarrow{\beta}^3| + |\overrightarrow{\tilde{\overline{y}}_{j-1}\tilde{\overline{z}}'_{j-1}}| < \frac{C|yz|\nu}{\delta R\Sigma_{j-1}(y)} + \varkappa(\delta)|yz|.$$

On the other hand, according to Corollary 2.5 and Lemma 2.4 it follows from (4.12) that

$$\overrightarrow{\tilde{y}_{j-1}\tilde{z}'_{j-1}} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{\tilde{\overline{y}}_{j-1}\tilde{\overline{z}}_{j-1}}.$$

Therefore we have

$$|\overrightarrow{\beta}^2| \leqslant \varkappa(\delta)|\overrightarrow{\tilde{\overline{y}}_{j-1}\tilde{\overline{z}}_{j-1}}| \leqslant \varkappa(\delta)\left(\frac{C|yz|\nu}{\delta R\Sigma_{j-1}(y)} + \varkappa(\delta)|yz|\right).$$

Now we can conclude that

$$|\overrightarrow{\beta}| \leqslant |\overrightarrow{\beta}^1| + |\overrightarrow{\beta}^2| + |\overrightarrow{\beta}^3| + |\overrightarrow{\beta}^4| < (1+\varkappa(\delta))|\overrightarrow{\alpha}_{j-1}| + \frac{C|yz|\nu}{\delta R\Sigma_{j-1}(y)} + \varkappa(\delta)|yz|.$$

And plugging the estimates of $|\overrightarrow{\beta}|$ and $|\overrightarrow{\gamma}|$ (see (4.11)) into (4.10), we obtain the **Subclaim** (and thus **the whole proof is completed**). □

5 Appendix

In Appendix, we give the proofs of (2.1.1), Lemma 2.4 and (4.5). In the proof of (2.1.1), we will use a result contained in Lemma 5.6 in [1].

Lemma 5.1 *Let $p, q, r, s \in M$. For sufficiently small δ, if $|qs| < \delta \cdot \min\{|pq|, |rq|\}$ and $\tilde{\angle}pqr > \pi - \delta$, then $|\tilde{\angle}pqs - \angle pqs$[7]$| < \varkappa(\delta)$ and $|\tilde{\angle}rqs - \angle rqs| < \varkappa(\delta)$.*

[6]When applying Corollary 2.5, we can assume that (s_i^{j-1}, t_i^{j-1}) is also an R-long $(n, 2\delta)$-strainer at $h_j(x_j)$ (see the beginning of the proof of (4.5) in Appendix).

[7]$\angle pqs$ is the angle between geodesics qp and qs at q, which is well defined by $\lim\limits_{x,y\longrightarrow q}\tilde{\angle}xqy$ with $x \in qp$ and $y \in qs$.

Proof of (2.1.1):

According to Lemma 5.1, (2.1.1) is equivalent to

$$|\angle a_iq_jr_j-\angle a_iq_{j'}r_{j'}|<\varkappa(\delta)\iff|\angle s_iq_jr_j-\angle s_iq_{j'}r_{j'}|<\varkappa(\delta)\text{ for }i=1,\cdots,n.\quad(5.1)$$

Using the law of cosine, it is not difficult to conclude

$$|\tilde\angle uq_jv-\tilde\angle uq_{j'}v|<\varkappa(\delta)\text{ for }u\in\{s_i,t_i\}_{i=1}^n\text{ and }v\in\{a_i,b_i\}_{i=1}^n.$$

By Lemma 5.1 again,

$$|\angle uq_jv-\angle uq_{j'}v|<\varkappa(\delta).\quad(5.2)$$

Now we consider spaces of directions at q_j, Σ_{q_j}, with angle metric. In the situation here, Theorem 9.5 in [1] ensures that Σ_{q_j} is $\varkappa(\delta)$-almost isometric to an $(n-1)$-dimensional unit sphere. Denote by $\bar a_i\in\Sigma_{q_j}$ (resp. $\bar s_i$ and $\bar r_j$) the directions of geodesics q_ja_i (resp. q_js_i and q_jr_j) for $i=1,\cdots,n$. Note that

$$|\bar a_i\bar a_{i'}|=\frac{\pi}{2}\pm\varkappa(\delta)\text{ and }|\bar s_i\bar s_{i'}|=\frac{\pi}{2}\pm\varkappa(\delta)\text{ for }i\neq i'.$$

Then it is not difficult to see that inequality (5.2) implies (5.1). □

Proof of Lemma 2.4:

We only give the proof for $k=0$ (proofs for other cases are similar). We first note that

$$\begin{aligned}&|\tilde\angle a_ix_1y_1-\tilde\angle a_ix_2y_2|<\varkappa(\delta)\\ \iff&|\cos\tilde\angle a_ix_1y_1-\cos\tilde\angle a_ix_2y_2|<\varkappa(\delta)\\ \iff&\left|\frac{|a_ix_1|^2+|x_1y_1|^2-|a_iy_1|^2}{2|a_ix_1|\cdot|x_1y_1|}-\frac{|a_ix_2|^2+|x_2y_2|^2-|a_iy_2|^2}{2|a_ix_2|\cdot|x_2y_2|}\right|<\varkappa(\delta)\\ \iff&\left|\frac{|a_ix_1|-|a_iy_1|}{|x_1y_1|}-\frac{|a_ix_2|-|a_iy_2|}{|x_2y_2|}\right|<\varkappa(\delta)\qquad(5.3)\\ \iff&\left|\frac{|a_ix_1|-|a_iy_1|}{|f(x_1)f(y_1)|}-\frac{|a_ix_2|-|a_iy_2|}{|f(x_2)f(y_2)|}\right|<\varkappa(\delta)\ (f\text{ is a }\varkappa(\delta)\text{-almost isometry}).\end{aligned}$$

Recall that $f(x)=(|a_1x|,|a_2x|,\cdots,|a_nx|)$. Hence $|\tilde\angle a_ix_1y_1-\tilde\angle a_ix_2y_2|<\varkappa(\delta)$ for $i=1,2,\cdots,n\iff\angle(\overrightarrow{f(x_1)f(y_1)},\overrightarrow{f(x_2)f(y_2)})<\varkappa(\delta)$. □

Proof of (4.5):

We only give the proof for $k=0$.

We first give an observation that $\{s_i^j,t_i^j\}_{i=1}^n$ is an R-long $(n,C\delta)$-strainer at any x_l for $l=1,\cdots,N_2$ (note that $|x_jx_l|\leqslant N_2R\delta\leqslant NR\delta$ with N depending only on n). Without loss of generality, we can assume that $\{s_i^j,t_i^j\}_{i=1}^n$ is an R-long (n,δ)-strainer at x_l, and thus $\{h(s_i^j),h(t_i^j)\}_{i=1}^n$ is an R-long $(n,2\delta)$-strainer at $h(x_l)$.

Next we note that inequality (4.5) is equivalent to for any $1\leqslant j,l_1,l_2\leqslant N_2$

$$|\tilde\angle h(a_i^j)h_{l_1}(y)h_{l_1}(z)-\tilde\angle h(a_i^j)h_{l_2}(y)h_{l_2}(z)|<\varkappa(\delta).$$

On the other hand, for $i = 1, \cdots, n$ and any $u \in \{s_i^j, t_i^j\}_{i=1}^n$

$$\begin{aligned}
& |\tilde{\angle} h(a_i^j)h_{l_1}(y)h_{l_1}(z) - \tilde{\angle} h(a_i^j)h_{l_2}(y)h_{l_2}(z)| < \varkappa(\delta) \\
\text{(by (2.1.1))} \iff & |\tilde{\angle} h(u)h_{l_1}(y)h_{l_1}(z) - \tilde{\angle} h(u)h_{l_2}(y)h_{l_2}(z)| < \varkappa(\delta) \\
\text{(obviously)} \Longleftarrow & |\tilde{\angle} h(u)h_l(y)h_l(z) - \tilde{\angle} uyz| < \varkappa(\delta) \text{ for } l = 1, \cdots, N_2 \\
\text{(by (5.1))} \iff & |\angle h(u)h_l(y)h_l(z) - \angle uyz| < \varkappa(\delta) \\
\text{(?)} \iff & |\angle h(a_i^l)h_l(y)h_l(z) - \angle a_i^l yz| < \varkappa(\delta) \qquad (5.4) \\
\text{(by Lemma 5.1)} \iff & |\tilde{\angle} h(a_i^l)h_l(y)h_l(z) - \tilde{\angle} a_i^l yz| < \varkappa(\delta) \\
\text{(see (5.3))} \iff & \left| \frac{|h(a_i^l)h_l(y)| - |h(a_i^l)h_l(z))|}{|h_l(y)h_l(z)|} - \frac{|a_i^l y| - |a_i^l z|}{|yz|} \right| < \varkappa(\delta),
\end{aligned}$$

where the last inequality holds because $|h(a_i^l)h_l(y)| = |a_i^l y|$ and $|h(a_i^l)h_l(z)| = |a_i^l z|$ (recall that $h_l = g_l^{-1} \circ f_l$), and h_l is a $\varkappa(\delta)$-almost isometry.

Hence we only need to verify the third ' $\iff$ ' in (5.4). Similar to getting inequality (5.2), we can obtain for any $v \in \{a_i^l, b_i^l\}_{i=1}^n$

$$|\angle h(u)h_l(y)h(v) - \angle uyv| < \varkappa(\delta).$$

Therefore we can use the same argument as the end of the proof of (2.1.1) to conclude the third ' $\iff$ ' in (5.4) holds (taking into account that both $\Sigma_{h_l(y)}$ and Σ_y are $\varkappa(\delta)$-almost isometric to $\mathbb{S}^{n-1}$). □

Acknowledgement: The authors are indebted to Professor Yuri Burago for his very precious suggestions on the notations, formulations and structure (and so on) of the present paper.

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